BREAKDOWN THEORY OF A GRANULAR BED BY THE FLOW OF A CONTINUUM

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The article discusses the statement of problems of local breakdown of a granular bed under the action of a stream filtering through it, and a general method of their approximate analysis is suggested. The simplest examples are examined.

The hydraulic forces originating in the filtration of a gas or dropping liquid in a granular bed have a considerable influence on the stress state and the structure of the bed. If the characteristic flow rate is sufficiently small, so that these forces are not large compared with the force of gravity, then such an influence is limited by the possibility of local repacking of particles leading to local changes in the effective permeability of the bed, which manifests itself in the distribution of the flow, and consequently, also of the hydraulic forces within it. When the mentioned critical speed, depending on the stress state of the bed (i.e., geometry of the bed, type of flow through it, the physical characteristics of the granular material and of the continuum) is attained, the limiting stress state is realized in the entire bed or within a certain part of it. When the rate is further increased, it is impossible to maintain the state of equilibrium of the immovable bed: plastic flow begins which leads to the formation of cavities in the bed, and possibly to the subsequent liquefaction of the granular material.

The practical importance of the processes of local breakdown of the granular bed is very great. These processes, in particular, determine the nature and the parameters of the initial development of spouts in the spouting layers [1] and altogether the form and dynamics of development of cavities upon introduction of jets of the medium into the granular bed [2]. In this connection such processes are also important to the formation of the structure of the zone adjacent to the screen of apparatus with fluidized or immovable infiltrated beds, especially in the usually used fairly coarse gas-distributing devices [3]. In particular, it is possible that situations occur with partial liquefaction of the granular material localized in limited zones near individual jets introduced into the bed by such a device. Foci of breakdown inside the bed may be caused by inhomogeneities of packing of the particles, and also by inhomogeneities of the stream, as is the case in processes of channel formation and in flow around various baffles and obstacles inside the bed [4, 5]. Finally, the same type of processes determines the initial liquefaction of granular beds even when the stream of liquefying agent is uniformly distributed.

Even if the distribution of the stream and of the hydraulic forces in the bed is known, the analysis of the above processes leads to very complicated problems where it is necessary to determine not only the critical parameters of the stream but also the shape of the region of plastic flow corresponding to incipient breakdown.* This complexity is to a large extent due to the lack of a completely satisfactory model of granular media, and also to mathematical difficulties arising in solving the quasilinear system of equations of static equilibrium, closed with the aid of some additional assumption, in the region with unknown boundary. The most natural hypothesis of closure consists in the assumption that immediately before breakdown within the entire mentioned region (and not only, let us say, on its borders) Coulomb's law is fulfilled. An alternative hypothesis, that in this region one of the principal stresses vanishes, was suggested by Gupalo and Cherepanov [9] and used, in particular, for describing in [10] the local liquefaction of the bed near baffles built into it. This hypothesis is apparently correct in liquefaction of beds with constant cross section by a uniform

*An exception are also the rather complex problems of the initial fluidization in apparatuses with vertical [6] and lightly inclined [7, 8] walls where it may be assumed that the boundaries of this region coincide with the walls, i.e., that they are previously known. However, with increasing angle of slope, such an assumption ceases to be correct: the initial breakdown of the bed occurs in a relatively narrow channel [1] inside the bed, with subsequent development of spouting.

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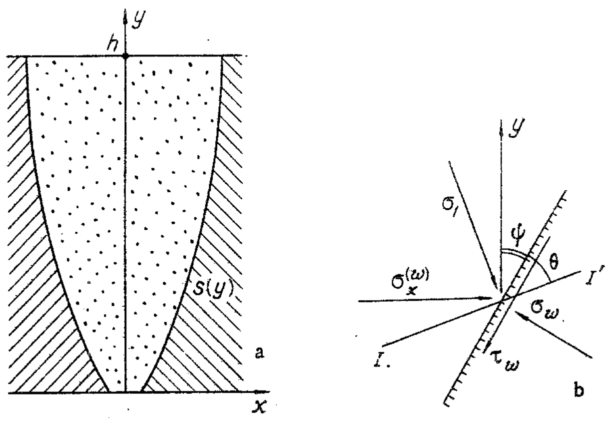

Fig. 1. Illustration of the region of plastic shear (a) and diagram of the stresses acting on its boundary (b).

stream, if the bed is free of spreading stresses and associated forces of near-wall friction. In more complicated situations, as indicated by the results of [8], breakdown of the bed occurs under conditions when none of the principal stresses inside the region of plastic shear vanishes. Below we use the first of the mentioned hypotheses with some additional simplifying assumptions.

First of all, we consider the distribution of the flow of continuum known from the solution of the corresponding filtration problem. Here we completely neglect the existence of feedback between the characteristics of the stream and the state of the granular material which we consider homogeneous and incompressible. That means that we neglect both the elastic and plastic deformations of the material under the effect of hydraulic forces, i.e., local porosity and permeability of the bed are thought not to be dependent on the parameters of the stream up to the instant of breakdown of static equilibrium. We note, however, that in many cases the correlation between the state of stress of the bed and its effective hydraulic characteristics may play an important part. This can explain, e.g., the different nature of the distribution of the gas stream over the cross section of the granular bed in filtration in the direction of the force of gravity and in the opposite direction [11].

For the sake of simplicity, we examine here only problems where there is symmetry with respect to a vertical plane or axis, although the methods used can easily be generalized to more complicated situations. Let the region of limiting plastic equilibrium, when the stream attains its critical value, have the shape illustrated in Fig. 1a. The horizontal coordinate x represents an ordinary Cartesian or radial coordinate for a plane or axisymmetric problem, respectively; the surface $|x| = s(y)$ bounds the mentioned region, and between the tangential and normal stresses on this surface, the following correlation applies:

$$\tau_w = \sigma_w \operatorname{tg} \varphi. \tag{1}$$

First we find the correlation between the magnitudes in (1) and the stresses in the examined system of coordinates at the boundary of the region. If ψ is the angle between the slope of the plane tangential to the boundary and the vertical, and θ is the angle between this plane and the area II' of action of the maximum principal compressive stress (Fig. 1b), then the state of the material at the examined point of the boundary is described by the points C, A, and A' on the Mohr circle shown in Fig. 2. It follows from an analysis of the Mohr circle that

$$\begin{gathered}\sigma_w = R \sin^{-1}\varphi \cos^2\varphi, \quad \tau_w = R\cos\varphi, \\ \sigma_{x,y}^{(w)} = R\sin^{-1}\varphi\,[1 \mp \sin\varphi \sin(2\psi + \varphi)], \\ \tau_{xy}^{(w)} = R\cos(2\psi + \varphi), \quad 2R = \sigma_1 - \sigma_2, \quad \psi = \operatorname{arctg} s'(y).\end{gathered} \tag{2}$$

To eliminate the necessity of investigating the state of stress inside the region in detail and to simplify considerably the calculations, it is desirable immediately to introduce, instead of the magnitudes $\sigma_{x,y}(x, y)$ and $\tau_{xy}(x, y)$, the corresponding magnitudes averaged over the sections of the region by the planes y = const. If the angle of near-wall friction is considerably smaller than the angle φ of internal friction of the granular material, it is often recommended to determine the approximate correlation between these magnitudes on the

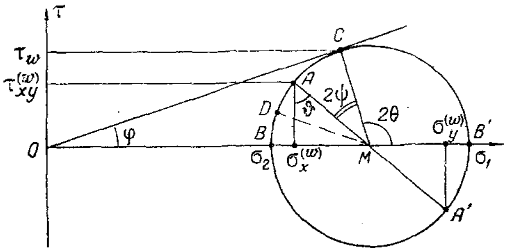

Fig. 2. Mohr circle for the limiting state of stress inside the region of plastic shear.

basis of the assumption that the horizontal normal stress over the section is constant [12, 13]. In the case under examination, the mentioned angles coincide, and such an assumption is fairly rough: it means that the state of stress at any point of the section is described by the same points on the Mohr circle as at the boundary points of the section, i.e., it corresponds to the Jansen model with a coefficient of lateral pressure depending on ψ, and consequently also on y, which can be easily expressed from (2) as the ratio of the stresses $\sigma_x^{(w)}/\sigma_y^{(w)}$.

In reality, when x changes from zero to s(y), point D of the Mohr circle in Fig. 2 moves monotonically upward from position B to its limiting position at point A, and the stresses σ_x, σ_y change correspondingly from σ_2, σ_1 to $\sigma_x^{(w)}$, $\sigma_y^{(w)}$. The simplest assumption refining Jansen's model corresponds to the hypothesis that the angle ϑ in Fig. 2 then changes approximately linearly from its limiting value $\pi/2$ for x = 0 to $2\psi + \varphi$ for x = s(y). Then the stresses $\sigma_{x,y}(x, y)$ are expressed by the same kind of formulas as $\sigma_{x,y}^{(w)}(x)$ in (2) if in the latter we replace $2\psi + \varphi$ by $\pi/2-(\pi/2-2\psi-\varphi)[x/s(y)]$. Therefore, for the plane problem we obtain

$$\langle \sigma_{x,y} \rangle = \frac{R}{\sin\varphi}\left[1 \mp \frac{\sin\varphi\cos(2\psi+\varphi)}{\pi/2-2\psi-\varphi}\right], \tag{3}$$

and for the axisymmetric problem

$$\langle \sigma_{x,y} \rangle = \frac{R}{\sin\varphi}\left\{1 \mp 2\sin\varphi\left[\frac{\cos(2\psi+\varphi)}{\pi/2-2\psi-\varphi}+\frac{\sin(2\psi+\varphi)-1}{(\pi/2-2\psi-\varphi)^2}\right]\right\}. \tag{4}$$

Taking (2) and (3) into account, we have for the plane problem ($\sigma \equiv \langle\sigma_y\rangle$)

$$\frac{\sigma_w}{\sigma} = \frac{(\pi/2-2\psi-\varphi)\cos^2\varphi}{\pi/2-2\psi-\varphi+\sin\varphi\cos(2\psi+\varphi)}, \tag{5}$$

and for the axisymmetric problem on the basis of (2) and (4)

$$\frac{\sigma_w}{\sigma} = \frac{(\pi/2-2\psi-\varphi)^2\cos^2\varphi}{(\pi/2-2\psi-\varphi)^2+2\sin\varphi\,[(\pi/2-2\psi-\varphi)\cos(2\psi+\varphi)+\sin(2\psi+\varphi)-1]}. \tag{6}$$

In both cases, τ_w is expressed by σ_w with the aid of (1).

Examining the balance of forces acting on the element (y, y + dy) of the investigated region, we write the equilibrium equation. We have (cf. [12, 13])

$$\begin{gathered}\frac{d(s^k\sigma)}{dy} + k(\tau_w - \sigma_w \operatorname{tg}\psi)s^{k-1} = f_k(y;\ s;\ u),\\ f_k(y;\ s;\ u) = k\int_0^s F(x,\ y;\ u)\,x^{k-1}\,dx - \gamma s^k,\end{gathered} \tag{7}$$

where k is equal to 1 and 2 for the plane and the axisymmetric problem, respectively. If we express τ_w and σ_w by σ, we obtain finally from (7) in accordance with (1), (5), and (6) that

$$\frac{d(s^k\sigma)}{dy} + A_k s^{k-1}\sigma = f_k(y;\ s;\ u), \tag{8}$$

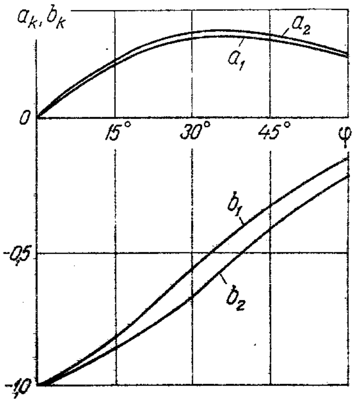

Fig. 3. Dependence of the coefficients a and b for the plane and the axisymmetric problems on the angle of internal friction of the granular medium.

where

$$A_1 = \frac{(\operatorname{tg}\varphi - \operatorname{tg}\psi)(\pi/2 - 2\psi - \varphi)\cos^2\varphi}{\pi/2 - 2\psi - \varphi + \sin\varphi\cos(2\psi + \varphi)},$$
$$A_2 = \frac{(\operatorname{tg}\varphi - \operatorname{tg}\psi)(\pi/2 - 2\psi - \varphi)^2\cos^2\varphi}{(\pi/2-2\psi-\varphi)^2+2\sin\varphi\,[(\pi/2-2\psi-\varphi)\cos(2\psi+\varphi)+\sin(2\psi+\varphi)-1]}. \tag{9}$$

The mathematical investigation of the problem is greatly complicated by the nonlinear dependence of A_k on $\psi = \arctan s'(y)$. On the other hand, in many practical situations, the angle of slope of the boundaries of the region to the vertical is small compared with the angle of internal friction of the material, and it is an advantage to use the Taylor expansion of A_k from (9)

$$A_k(\varphi,\ \psi) = a_k(\varphi) + b_k(\varphi)\,s'(y) + \ \ldots,\ \ s'(y) = \operatorname{tg}\psi \tag{10}$$

(the analytical expressions of the coefficients a_k, b_k, . . . are not written out because they are so long). Summary final results may be obtained on the basis of models in which the first one or two terms of the series (10) are taken into account. In that case it is not clear beforehand which of these two models is preferable, and the choice between them has to be made by comparison with the experiment. The dependences of the coefficients a_k, b_k on φ are shown in Fig. 3.

The general solution of Eq. (8), with the obvious boundary condition of σ vanishing at the level $y = 0$, is written in the form

$$\sigma(y;\ s;\ u) = \frac{e^{-G}}{s^k}\int_0^y f e^G\,dy,\quad G(y;\ s) = \int_0^y \frac{a + bs'}{s}\,dy, \tag{11}$$

where the functions f, a, and b are determined in (7), (9), and (10), and the subscript k is omitted for simplicity; the first of the above models corresponds to $b = 0$. The equation for determining the critical flow rate, corresponding to the beginning of breakdown of the bed, is obtained in implicit form from the requirement that the stress vanishes also at the upper boundary $y = h$ of the examined region [8], i.e.,

$$\sigma(h;\ s;\ u) = 0. \tag{12}$$

The value of u hence determined represents the functional of the functions s(y) whose concrete form depends on the distribution of the stream of the continuum in the system and on the type of dependence of the specific hydraulic force F on its velocity. However, in the most general case it may be assumed that the bed breaks down when the minimum velocity u_m is attained at which in principle it is possible that a region of plastic flow, emerging on the upper boundary of the bed (cf. [8]), forms. Thus, we arrive at the following extremum problem: we have to find such a profile $s_m(y)$ of the mentioned region that the functional u, implicitly determined by the relation (12), assumes its minimum value; then the corresponding value u_m is also calculated from (12) by substituting the function $s_m(y)$, already found, into the integral in (11).

By a number of perfectly natural simplifications, the formulated problem is reduced to the standard problems of variational calculus. Thus, we describe the force F with the aid of the exponential law*

$$F(x,\ y;\ u) = \alpha u^m V^{m-1}(x,\ y) V_y(x,\ y), \tag{13}$$

where V is the dimensionless speed of filtering. Then the function f, which also figures in (11), is represented in the form

$$f = \alpha u^m v^{(m)} - \gamma s^k,\ v^{(m)}(y;\ s) = k\int_0^s V^{m-1} V_y x^{k-1}\, dx. \tag{14}$$

In addition, we introduce the function

$$g(y) = \exp\left(a\int_0^y \frac{dy}{s}\right),\quad e^G = \left(\frac{s}{s_0}\right)^b g, \tag{15}$$

where s_0 is the value of s(y) when y = 0. Then it follows from (11), (12), (14), and (15) that

$$s = a\frac{g}{g'},\quad u = \left(\frac{\gamma}{\alpha}\ \frac{\Phi_1}{\Phi_2}\right)^{1/m}, \tag{16}$$

where

$$\Phi_1(g, g') = \int_0^h s^{k+b} g\, dy,\ \Phi_2(g,\ g') = \int_0^h v^{(m)} s^b g\, dy. \tag{17}$$

If $\Phi_1 = C = \text{const}$, then the examined problem reduces to the isometric problem of the maximum of the functional Φ_2. The sought extrema will be the unconditional extrema of the functional $\Phi = \Phi_2 + \lambda\Phi_1$, where λ is the Lagrange multiplier which can be determined a posteriori from the condition $\Phi_1 = C$ [14]. Obviously, when y = 0, we must have g = 1; when y = h, the "natural" boundary condition [14] has to be satisfied. Introducing the function

$$H(y;\ s;\ \lambda) = [v^{(m)}(y;\ s) + \lambda s^k]\, s^b, \tag{18}$$

where s(y) is expressed through g(y), and its derivative with the aid of the first ratio in (16), we write the Euler equation for the functional Φ and the boundary conditions to it. Taking into account the equalities

$$\frac{\partial H}{\partial g} = \frac{a}{g'}\frac{\partial H}{\partial s},\quad \frac{\partial H}{\partial g'} = -\frac{ag}{g'^2}\frac{\partial H}{\partial s} \tag{19}$$

and using the determination of Φ, we obtain after calculations that

$$\frac{d}{dy}\left(s^2\frac{\partial H}{\partial s}\right) + a\left(s\frac{\partial H}{\partial s} + H\right) = 0,\quad \left.\frac{\partial H}{\partial s}\right|_{y=h} = 0. \tag{20}$$

The solution of this problem yields the function s(y) which depends additionally on λ or C as on a parameter; the corresponding function g(y) is calculated from the solution of the first equation in (16) on condition that g(0) = 1. After this it is easy to calculate the functionals Φ_1 and Φ_2 from (17) and to determine u from the second ratio in (16). Then we have to compare the values u corresponding to different λ, and select the minimal u_m from them corresponding to some λ_m. The value u_m and the solution $s_m(y)$ of the problem (20), with $\lambda = \lambda_m$ provide the solution of the problem.

The described calculations are very laborious, and in most cases they can be carried out only numerically. Therefore, to obtain an approximate solution in analog form, it is advisable to use strong direct methods of variational calculus [14] and be guided in the selection of the sample functions s(y) by experimental data or by some probable physical considerations.

*Any other empirical formula may also be used for the force, e.g., Ergan's binomial formula, but this leads to more cumbersome calculations. We also note that according to the equations of the theory of filtration, the force F may be replaced by $(K/\mu)(\partial p/\partial y)$, where p is the pressure; μ is the viscosity of the continuum; K is the permeability of the bed.

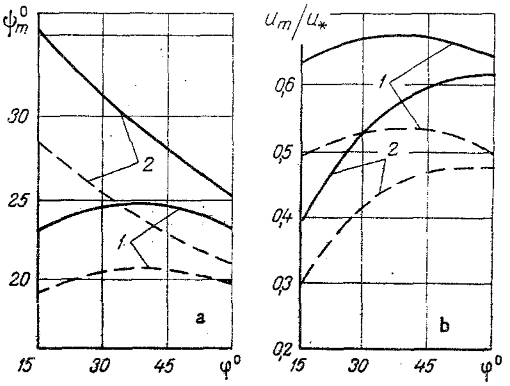

Fig. 4. Dependence of the angle of slope ψ_m^0 of the boundary of the region of plastic shear (a) and of the parameter u_m/u_*, corresponding to the beginning of breakdown (b), on φ for plane and axisymmetric jets (solid and dashed lines, respectively); curves 1 and 2 correspond to the models with $A \approx a$ and $A \approx a + b\tan\psi$.

As examples, which even by themselves are of interest, we will examine the puncture of a fine-grained bed (m = 1) by a plane and an axisymmetric stream. For the sake of simplicity we assume, as in [7, 8], that the streams diverge radially, i.e.,

$$u = \frac{q}{(h+y_0)^k}, \quad V_y = \frac{(h+y_0)^k (y+y_0)}{[(y+y_0)^2 + x^2]^{(1+k)/2}}. \tag{21}$$

Equations (21) are strictly fulfilled for pan-type and conical apparatuses when the lower and upper boundaries of the bed are cylindrical or spherical surfaces with radii y_0 and $h + y_0$, respectively; for beds of other shape, e.g., for an unbounded horizontal bed, (21) is not correct near the mouth of the jet as well as in the vicinity of the upper boundary of the bed. In this problem, the magnitude q has the meaning of flow rate of the medium per unit linear or solid angle, and u, the speed of the medium at the upper boundary of the bed.

We consider the channel, over which the breakdown of the bed is effected first, as fairly narrow, and we expand V_y into a series of powers of $x/(y + y_0)$. Then, in accordance with (13) and (14), with k = 1 and 2, we obtain from (21)

$$\begin{aligned} v^{(1)}\Big|_{k=1} &\approx s\,\frac{h+y_0}{y+y_0}\left[1-\frac{1}{3}\left(\frac{s}{y+y_0}\right)^2\right], \\ v^{(1)}\Big|_{k=2} &\approx s^2\left(\frac{h+y_0}{y+y_0}\right)^2\left[1-\frac{3}{4}\left(\frac{s}{y+y_0}\right)^2\right]. \end{aligned} \tag{22}$$

In case of radial flow of the jet, it is natural to expect that s(y) is a linear function of $y + y_0$. There is also experimental evidence available that s(y) is close to linear even in more general situations [2, 8]. We therefore approximate

$$s(y) = \nu^{-1}(y + y_0), \tag{23}$$

where ν is a parameter that has to be determined.

If we calculate the functional (17) with the aid of (22) and (23), we obtain from the second equation in (16) for a plane jet

$$u = \frac{u_*}{1+z}\,\frac{\nu^2}{\nu^2 - 1/3}\,\frac{1+b+a\nu}{2+b+a\nu}\,\frac{(1+z)^{2+b+a\nu} - z^{2+b+a\nu}}{(1+z)^{1+b+a\nu} - z^{1+b+a\nu}} \tag{24}$$

and for an axisymmetric jet

$$u = \frac{u_*}{(1+z)^2}\,\frac{\nu^2}{\nu^2 - 3/4}\,\frac{1+b+a\nu}{3+b+a\nu}\,\frac{(1+z)^{3+b+a\nu} - z^{3+b+a\nu}}{(1+z)^{1+b+a\nu} - z^{1+b+a\nu}}. \tag{25}$$

Here we introduced the minimum rate of fluidization u_* and the relative size of the hole z through which the jet flows out:

$$u_* = \gamma/\alpha, \quad z = y_0/h. \tag{26}$$

For the sake of simplicity, we examine only the situation when $z \ll 1$. Then the value ν_m, providing the minimum for the expression (24), satisfies the equation

$$a\nu_m\left(\nu_m^2 - \frac{1}{3}\right) = \frac{2}{3}(1 + b + a\nu_m)(2 + b + a\nu_m), \tag{27}$$

and the unique minimum of (25) is attained for $\nu = \nu_m$, where ν_m is the root of the equation

$$a\nu_m\left(\nu_m^2 - \frac{3}{4}\right) = \frac{3}{4}(1 + b + a\nu_m)(3 + b + a\nu_m). \tag{28}$$

The dependences of the corresponding angle between the slope $\psi_m = \arctan \nu_m^{-1}$ of the boundaries of the region of plastic flow and the vertical on the angle of internal friction of the granular medium corresponding to both suggested models ($b = 0$ and $b < 0$) are shown in Fig. 4a. As was to be expected on the basis of general considerations, for an axisymmetric jet, the values of ψ_m are substantially smaller than for the plane jet. Figure 4b shows the corresponding dependences of φ on the magnitude of u_m/u_*. This magnitude is smaller than unity, i.e., breakdown of the bed begins at linear filtration velocities on the upper boundary that are smaller than the initial speed of fluidization [7, 8].

The curves in Fig. 4 favor selection of the first model when the dependence of A on φ is not taken into account at all. In fact, this model correctly expresses the increase in u_m/u_* with increasing ψ_m in accordance with the experimental data presented, e.g., in [2, 8]. The second model describes the correlation between u_m/u_* and ψ_m incorrectly. This is not surprising because the angle ψ_m is approximately the same as φ, so that using two terms of the expansion in (10) in this case is incorrect. However, the second model may be more accurate in situations where $\psi_m \ll \varphi$, i.e., for very narrow regions of plastic shear originating, e. g., during the process of channel formation.

We want to point out that if a bed is examined in an apparatus with a small angle between the walls, and we compare the value of u_*/u_m, calculated above, with the critical value determined in [8], we can answer the question of what does occur in reality – plastic shear of the examined type or shear along the walls of the apparatus.

The question of what happens to the system after the initial breakdown of static equilibrium requires independent analysis. Experience indicates that with further increase in the flow rate of the medium, a stable state of the bed with cavity is established; the height of the cavity increases with increasing rate until the critical flow rate is attained at which an instantaneous "puncture" of the bed occurs. An analysis of these phenomena, and also a comparison with experimental data, will be presented in one of the subsequent works.

NOTATION

A	is the function introduced into (9);
a, b	are the coefficients of expansion into Taylor series;
F	is the specific force of hydraulic resistance of the granular bed;
G, g	are the functions introduced into (11) and (15);
H	is the function from (18);
h	is the height of the bed;
k = 1 and 2	are the plane and axisymmetric problems, respectively;
m	is the coefficient in the power law for hydraulic resistance;
q	is the flow rate of the continuum per unit linear or solid angle;
R	is the radius of the Mohr circle;
s	is the coordinate of the boundary of the region of plastic shear;
u	is the dimensional filtration velocity;
u_*	is the minimum fluidization velocity;
$v^{(m)}$	is the function introduced into (14);
x, y	are the coordinates;

y_0, z	are the parameters determined in (21) and (26);
α	is the coefficient of hydraulic resistance;
γ	is the effective specific weight of the granular material;
θ	is the angle between the tangential plane on the boundary of the region of plastic shear and the area of maximum compressive stress;
ϑ	is the angle determined in Fig. 2;
λ	is the Lagrange multiplier;
ν	is the parameter in (23);
σ	is the normal compressive stress;
σ_1, σ_2	are the principal compressive stresses;
τ	is the tangential stress;
Φ_i	are the functionals introduced into (17);
φ	is the angle of internal friction of granular material;
ψ	is the angle between the plane, tangential to the boundary of the region of plastic shear, and the vertical.

Subscripts

m	relates to magnitudes corresponding to the initial plastic shear;
w	relates to the state of stress on the boundary of the region of plastic shear;
prime	differentiation of a function with respect to its argument;
angle brackets	averaged stresses with respect to the section y = const in the region of plastic shear.

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